MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes X

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Question

(Q1) Let $t = \tan\dfrac{x}{2}$. Show that

$$\int \frac{dx}{\left(5 + 4\cos x\right)\sin x} = \int \frac{1+t^2}{t^3+9t}dt.$$

Hence, evaluate the integral.

(Q2) Evaluate

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^4}\int_0^x \left(e^{t^2}-1\right)\ln\left(1+t\right)dt\right)$

(b) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^4}\int_0^x \left(e^{t^2}-1\right)\ln\left(1+x\right)dt\right)$

(c) $F'(e)$ given that $F(x) = \displaystyle\int_1^{x^2} \left(1+\ln t\right)^4 dt$

(Q3)(a) Let $a > 0$, and $f : [0, a] \to \mathbb{R}$ be a continuous function on $[0, a]$. Show that

$$\int_0^a f(x)dx = \frac{1}{A}\int_0^a \left(f\left(x\right) + f\left(a-x\right)\right)dx.$$

Here A is an integer whose value you have to determine explicitly.

(b) Hence, or otherwise, evaluate

$$\int_0^1 \frac{dx}{\left(x^2-x+1\right)\left(e^{2x-1}+1\right)}.$$

(Q4) Define the function $f : \left(0, \dfrac{\pi}{2}\right) \to \mathbb{R}$ by

$$f(x) = \int_1^x \sin\left(\cos t\right)dt$$

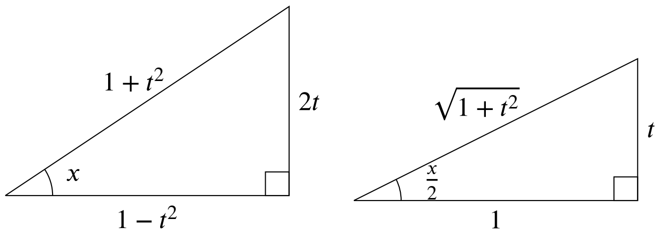
for any $x \in \left(0, \dfrac{\pi}{2}\right)$.

(a) Show that f is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$.

(b) Find all the value α such that $f(\alpha) = 0$. Justify your answer.

(c) Suppose g is the inverse of f. Find the value of $g'(0)$.

(A1) Let $t = \tan\dfrac{x}{2}$, then $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2t}{1-t^2}$.



By drawing two triangles (with angle x and $\dfrac{x}{2}$), we know

$$\begin{aligned}
\sin x &= \frac{2t}{1+t^2}\\
\cos x &= \frac{1-t^2}{1+t^2}\\
\tan x &= \frac{2t}{1-t^2}\\
dt &= \frac{1}{2}\sec^2\frac{x}{2}dx \iff dx = \frac{2dt}{1+t^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{dx}{\left(5+4\cos x\right)\sin x} &= \int \frac{\frac{2}{1+t^2}}{\left(5+4\cdot\frac{1-t^2}{1+t^2}\right)\cdot\frac{2t}{1+t^2}}dt\\
&= \int \frac{1+t^2}{t\left(9+t^2\right)}dt\\
&= \int \frac{1+t^2}{t^3+9t}dt.
\end{aligned}$$

Write $\dfrac{1+t^2}{t^3+9t} = \dfrac{A}{t} + \dfrac{Bt+C}{9+t^2} = \dfrac{(A+B)\,t^2+Ct+9A}{t^3+9t}$.

Compare the coefficient of x^2, x, constant term, then $\begin{cases} A+B &= 1\\ C &= 0\\ 9A &= 1\end{cases}$.

Hence, $A = \dfrac{1}{9}, B = \dfrac{8}{9}, C = 0$. Then

$$\begin{aligned}
\int \frac{dx}{\left(5+4\cos x\right)\sin x} &= \int \frac{1+t^2}{t^3+9t}dt\\
&= \int \left(\frac{1}{9t} + \frac{8t}{9\left(9+t^2\right)}\right)dt\\
&= \frac{1}{9}\int\frac{dt}{t} + \frac{4}{9}\int\frac{d\left(9+t^2\right)}{9+t^2}\\
&= \frac{\ln|t| + 4\ln\left|9+t^2\right|}{9} + C\\
&= \frac{\ln\left|\tan\frac{x}{2}\right| + 4\ln\left(9+\tan^2\frac{x}{2}\right)}{9} + C.
\end{aligned}$$

(A2)(a) Note as $x \to 0$, we have $\displaystyle\int_0^x \left(e^{t^2}-1\right)\ln\left(1+t\right)dt$ and $x^4 \to 0$, Note $\left(e^{t^2}-1\right)\ln\left(1+t\right)$ is continuous near $t=0$,

hence $\dfrac{d}{dx}\displaystyle\int_0^x \left(e^{t^2}-1\right)\ln\left(1+t\right)dt = \left(e^{x^2}-1\right)\ln\left(1+x\right)$ by Fundamental Theorem of Calculus.

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{x^4}\int_0^x \left(e^{t^2}-1\right)\ln\left(1+t\right)dt\right)\\
\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}}\ &\lim_{x\to 0}\frac{\left(e^{x^2}-1\right)\ln\left(1+x\right)}{4x^3}\\
=\ &\frac{1}{4}\lim_{x\to 0}\frac{e^{x^2}-1}{x^2}\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}\\
=\ &\frac{1}{4}.
\end{aligned}$$

(b) Using similar skill of (a), (try to write down the detail!) then

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{x^4}\int_0^x \left(e^{t^2}-1\right)\ln\left(1+x\right)dt\right)\\
&=\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}\lim_{x\to 0}\left(\frac{1}{x^3}\int_0^x \left(e^{t^2}-1\right)dt\right)\\
&=\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}\lim_{x\to 0}\frac{e^{x^2}-1}{3x^2}\\
&=\frac{1}{3}.
\end{aligned}$$

(c) Note that $\left(1+\ln t\right)^4$ is continuous on $\mathbb{R}$.

Then using Chain Rule and Fundamental Theorem of Calculus, we have

$$\begin{aligned}
F'(x) &= \left(\frac{d}{dx}x^2\right)\left(\frac{d}{dx^2}\int_1^{x^2}\left(1+\ln t\right)^4 dt\right)\\
&= 2x\left(1+\ln x^2\right)^4\\
&= 2x\left(1+2\ln x\right)^4.
\end{aligned}$$

Hence, $F'(e) = 2e\left(1+2\cdot 1\right)^4 = 162e$.

(A3)(a) Let $I=\displaystyle\int_0^a f\left(x\right)dx$.

Let $u=a-x$, that is $x=a-u$, then $du=-dx$,

when $x=0$, $u=a$ and when $x=a$, $u=0$. Then

$$I=\int_0^a f\left(x\right)dx=\int_a^0 f\left(a-u\right)\left(-du\right)=\int_0^a f\left(a-u\right)du=\int_0^a f\left(a-x\right)dx$$

Hence,

$$\begin{aligned}
2I = I+I &= \int_0^a f\left(x\right)dx+\int_0^a f\left(a-x\right)dx=\int_0^a \left(f\left(x\right)+f\left(a-x\right)\right)dx\\
I &= \frac{1}{2}\int_0^a \left(f\left(x\right)+f\left(a-x\right)\right)dx.
\end{aligned}$$

(Here, $A=2$.)

(b) Let $f(x)=\dfrac{1}{\left(x^2-x+1\right)\left(e^{2x-1}+1\right)}$.

Note f is continuous on $[0,1]$ with

$$\begin{aligned} f(x)+f(1-x) &= \frac{1}{\left(x^2-x+1\right)\left(e^{2x-1}+1\right)} + \frac{1}{\left(1-x+x^2\right)\left(e^{1-2x}+1\right)} \\ &= \frac{2+e^{2x-1}+e^{1-2x}}{\left(x^2-x+1\right)\left(2+e^{2x-1}+e^{1-2x}\right)} \\ &= \frac{1}{x^2-x+1}. \end{aligned}$$

By (a) and using the skill of tutorial IX, (try to write down the detail!) we have

$$\begin{aligned} \int_0^1 \frac{dx}{\left(x^2-x+1\right)\left(e^{2x-1}+1\right)} &= \frac{1}{2}\int_0^1 \frac{dx}{x^2-x+1} \\ &= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_{x=0}^{x=1} \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) \\ &= \frac{\pi}{3\sqrt{3}} \end{aligned}$$

(A4)(a) Note that $\sin(\cos t)$ is continuous on $\mathbb{R}$.

By Fundamental Theorem of Calculus, we have $f'(x)=\sin(\cos x)$.

If $x \in \left(0, \frac{\pi}{2}\right)$, we have $0 < \cos x < 1$, and so $f'(x) > 0$.

Then f is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

(b) Note $f(1) = \int_1^1 \sin(\cos t)\,dt = 0$.

By (a), f is strictly increasing, then $f(t) > f(1) = 0$ for any $t > 1$ and $f(t) < f(1) = 0$ for any $t < 1$.

Hence, there are unique root of f which is 1.

(c) By (b), we know $g(0) = 1$. Note

$$\begin{aligned} x &= g(y) \\ f(x) &= y \\ f'(x)\frac{dx}{dy} &= 1 \\ \frac{dx}{dy} &= \frac{1}{f'(x)} \end{aligned}$$

$$g'(0) = \left.\frac{dx}{dy}\right|_{y=0} = \left.\frac{dx}{dy}\right|_{x=1} = \frac{1}{f'(1)} = \frac{1}{\sin\left(\cos 1\right)} = \csc\left(\cos 1\right).$$